# A NEW LOOK

## at the

# STATE REPRESENTATION PROBLEM

**Jan C. Willems**
University of Groningen, NL
willems@math.rug.nl
http://www.math.rug.nl/∼willems/

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# OUTLINE

1. **Behavioral systems**
2. **State systems**
3. **Linear differential systems**
4. **State representations**
5. **State construction**
6. **Balanced realizations**
7. **State for PDE's**
8. **Conclusions**

# BEHAVIORAL SYSTEMS

## THE BASIC CONCEPT

***A dynamical system*** **=** $\boxed{\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})}$

$\mathbb{T} \subseteq \mathbb{R}$**, the *time-axis* (= the relevant time instances),**

$\mathbb{W}$**, the *signal space* (= where the variables take on their values),**

$\boxed{\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$ **: *the behavior***$}$ **(= the admissible trajectories).**

$$\boxed{\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})}$$

**For a trajectory $w : \mathbb{T} \to \mathbb{W}$, we thus have:**

**$w \in \mathfrak{B}$ : the model allows the trajectory $w$,**

**$w \notin \mathfrak{B}$ : the model forbids the trajectory $w$.**

**Usually, $\mathbb{T} = \mathbb{R}$, or $[0, \infty)$ (in continuous-time systems),**
**or $\mathbb{Z}$, or $\mathbb{N}$ (in discrete-time systems).**

**Usually, $\mathbb{W} \subseteq \mathbb{R}^{\mathtt{w}}$ (in lumped systems);**
**a function space**
**(in distributed systems, with time a distinguished variable);**
**or a finite set (in DES).**

**Emphasis today:** $\mathbb{T} = \mathbb{R}$, $\mathbb{W} = \mathbb{R}^{\mathtt{w}}$,
**$\mathfrak{B}$ = solutions of system of linear constant coefficient ODE's.**

## EXAMPLES

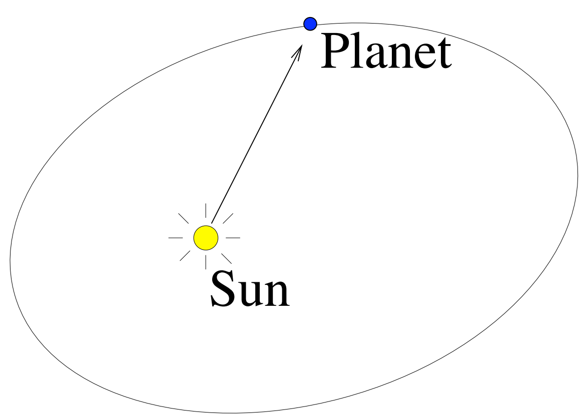
### 1. *Planetary orbits*

$\mathbb{T} = \mathbb{R}$ **(time),**

$\mathbb{W} = \mathbb{R}^3$ **(position),**

$\mathfrak{B}$ = **planetary orbits** $\cong$ **Kepler's laws:**

$$\text{ellipses, = areas in = time, } \frac{(\text{period})^2}{(\text{axis})^3} = \text{constant.}$$




**Planetary orbits**

## 2. *Input / output systems*

$$f_1\left(y(t), \frac{d}{dt}y(t), \frac{d^2}{dt^2}y(t), \ldots, t\right)$$

$$= f_2\left(u(t), \frac{d}{dt}u(t), \frac{d^2}{dt^2}u(t), \ldots, t\right)$$

$\mathbb{T} = \mathbb{R}$ **(time),**

$\mathbb{W} = \mathbb{U} \times \mathbb{Y}$ **(input $\times$ output signal spaces),**

$\mathfrak{B} =$ **all input / output pairs.**

## 3. *Codes*

$\mathbb{A}$ = the code alphabet, say, $\mathbb{A} = \mathbb{F}^{\mathtt{w}}$, $\mathbb{F}$ a finite field,

$\mathbb{I}$ = an index set, say,

$$\mathbb{I} = (1, \cdots, \mathtt{n}) \text{ in block codes,}$$
$$\mathbb{I} = \mathbb{N} \text{ or } \mathbb{Z} \text{ in convolutional codes,}$$

$\mathfrak{C} \subseteq \mathbb{A}^{\mathbb{I}}$ = the code; yields the system $\Sigma = (\mathbb{I}, \mathbb{A}, \mathfrak{C})$.

Redundancy structure, error correction possibilities, etc., are visible in the code behavior $\mathfrak{C}$.

It is the central object of study.

Encoder& decoder can be put (temporarily) into the background.

## 4. *Formal languages*

$\mathbb{A}$ = a (finite) alphabet,

$\mathfrak{L} \subseteq \mathbb{A}^*$ = the language = all 'legal' 'words' $a_1 a_2 \cdots a_k \cdots$
yields the system $\Sigma = (\mathbb{N}, \mathbb{A}, \mathfrak{L})$.

$\mathbb{A}^*$ = all finite strings with symbols from $\mathbb{A}$.

Examples: All words appearing in the *van Dale*
All LaTeX documents

# LATENT VARIABLE SYSTEMS

***First principles models*** $\rightsquigarrow$

*A dynamical system with latent variables* **=** $\boxed{\Sigma_L = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathfrak{B}_{\text{full}})}$

$\mathbb{T} \subseteq \mathbb{R}$, **the *time-axis* (= the set of relevant time instances).**
$\mathbb{W}$, **the *signal space* (= the variables that the model aims at).**

$\mathbb{L}$, **the *latent variable space* (= the auxiliary modeling variables).**

$$\boxed{\mathfrak{B}_{\text{full}} \subseteq (\mathbb{W} \times \mathbb{L})^{\mathbb{T}} : \textit{the full behavior}}$$

**(= the pairs $(w, \ell) : \mathbb{T} \to \mathbb{W} \times \mathbb{L}$ that the model declares possible).**

# THE MANIFEST BEHAVIOR

**Call the elements of** $\mathbb{W}$ ***'manifest' variables*** **,**

**those of** $\mathbb{L}$ ***'latent' variables*** **.**

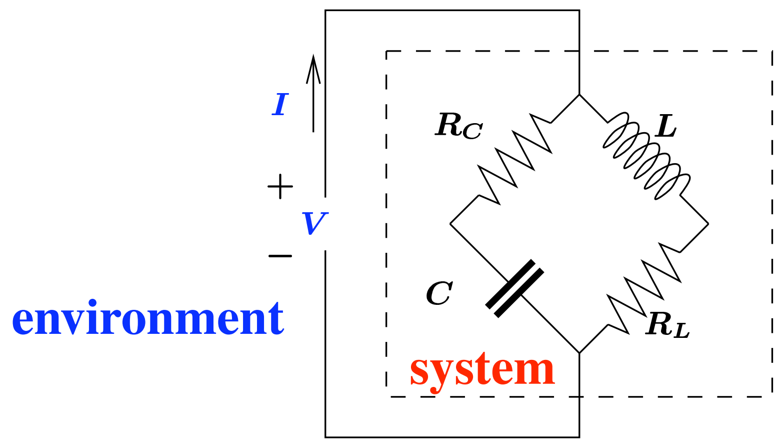
**The latent variable system** $\Sigma_L = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathfrak{B}_{\text{full}})$ **induces the** ***manifest system*** $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ **with** ***manifest behavior***

$$\mathfrak{B} = \{w : \mathbb{T} \to \mathbb{W} \mid \exists\ \ell : \mathbb{T} \to \mathbb{L} \text{ such that } (w, \ell) \in \mathfrak{B}_{\text{full}}\}$$

**In convenient equations for** $\mathfrak{B}$**, the latent variables are** ***'eliminated'*****.**

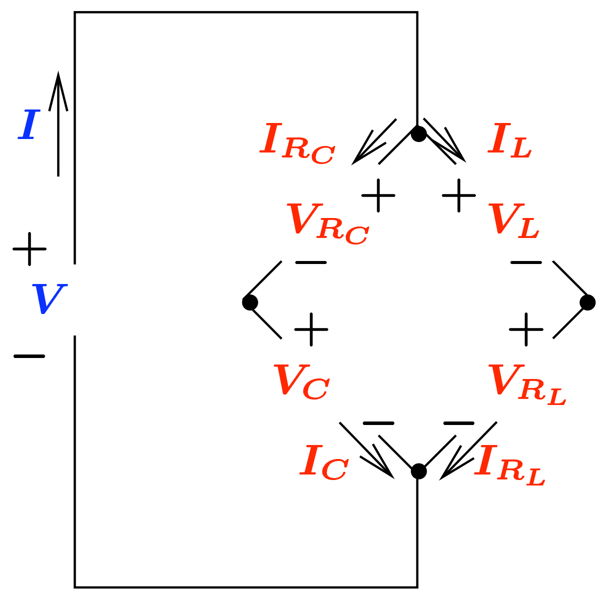
## EXAMPLES

### 1. *RLC - circuit*




**RLC - circuit**

**!! Model the relation between $V$ and $I$ !!**




**The circuit graph**

# SYSTEM EQUATIONS

**Introduce the following additional variables:**

**the voltage across and the current in each branch:**

$V_{R_C}, I_{R_C}, V_C, I_C, V_{R_L}, I_{R_L}, V_L, I_L.$

**Constitutive equations (CE):**

$$V_{R_C} = R_C I_{R_C}, \quad V_{R_L} = R_L I_{R_L}, \quad C\frac{d}{dt}V_C = I_C, \quad L\frac{d}{dt}I_L = V_L$$

**Kirchhoff's voltage laws (KVL):**

$$V = V_{R_C} + V_C, \quad V = V_L + V_{R_L}, \quad V_{R_C} + V_C = V_L + V_{R_L}$$

**Kirchhoff's current laws (KCL):**

$$I = I_{R_C} + I_L, \quad I_{R_C} = I_C, \quad I_L = I_{R_L}, \quad I_C + I_{R_L} = I$$

**Formalization as a latent variable system:**

$\mathbb{T} = \mathbb{R}$,
$\mathbb{W} = \mathbb{R}^2$ **- manifest variables: the port voltage and current,**
$\mathbb{L} = \mathbb{R}^8$ **- latent variables: the branch voltages and currents,**
$\mathfrak{B}_{\text{full}} =$ **all functions** $(V, I, V_{R_C}, I_{R_C}, V_C, I_C, V_{R_L}, I_{R_L}, V_L, I_L)$ **that satisfy the CE's, KCL, and KVL,**
$\mathfrak{B} =$ **the functions** $(V, I)$ **that satisfy the 'eliminated' port equations.**

## RELATION BETWEEN $V$ and $I$

After some calculations 'elimination', we obtain the port equations:

Case 1: $CR_C \neq \frac{L}{R_L}$.

$$\left(\frac{R_C}{R_L} + (1 + \frac{R_C}{R_L})CR_C\frac{d}{dt} + CR_C\frac{L}{R_L}\frac{d^2}{dt^2}\right)V = (1 + CR_C\frac{d}{dt})(1 + \frac{L}{R_L}\frac{d}{dt})R_C I.$$

Case 2: $CR_C = \frac{L}{R_L}$.

$$\left(\frac{R_C}{R_L} + CR_C\frac{d}{dt}\right)V = (1 + CR_C\frac{d}{dt})R_C I$$

These are the exact relations between $V$ and $I$ !

## 2. *Input /state / output systems*

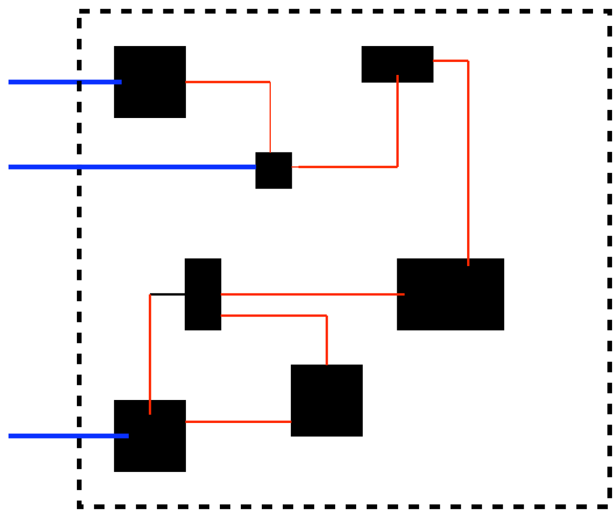
$$\frac{d}{dt}x(t) = f(x(t), u(t)); \quad y(t) = h(x(t), u(t)),$$

$\mathbb{T} = \mathbb{R}$, $\mathbb{W} = \mathbb{U} \times \mathbb{Y}$, $\mathbb{L} = \mathbb{X}$,

$\mathfrak{B}_{\text{full}} =$ **all** $(u, y, x) : \mathbb{R} \to \mathbb{U} \times \mathbb{Y} \times \mathbb{X}$ **that satisfy these equations,**

$\mathfrak{B} =$ **all (input / output)-pairs.**

## 3. *Interconnected systems* $\cong$ 'First principles' models



**Interconnected system**

**External variables** $\rightsquigarrow$ **manifest;**

**Internal variables** $\rightsquigarrow$ **latent.**

## 4. *Grammars*

**A convenient way to specify a formal language, whose essence is captured by latent variables, is through *grammars*.**

# STATE SYSTEMS

# THE NOTION OF STATE

**The latent variable system**

$$\Sigma_X = (\mathbb{T}, \mathbb{W}, \mathbb{X}, \mathfrak{B}_{\text{full}})$$

**is said to be a *state system* if**

$$(w_1, x_1), (w_2, x_2) \in \mathfrak{B}_{\text{full}} \quad \text{and} \quad x_1(t_0) = x_2(t_0)$$

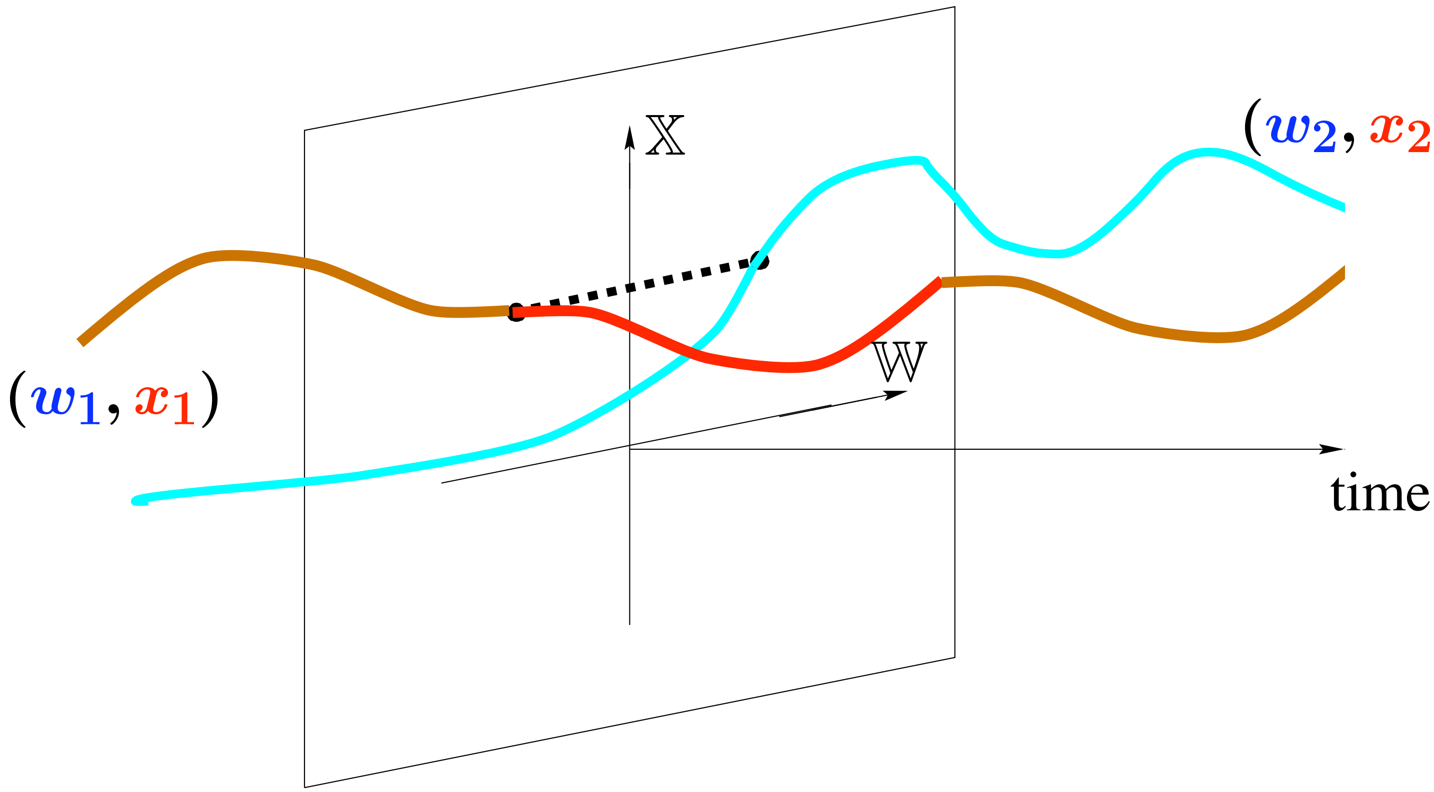
**imply**

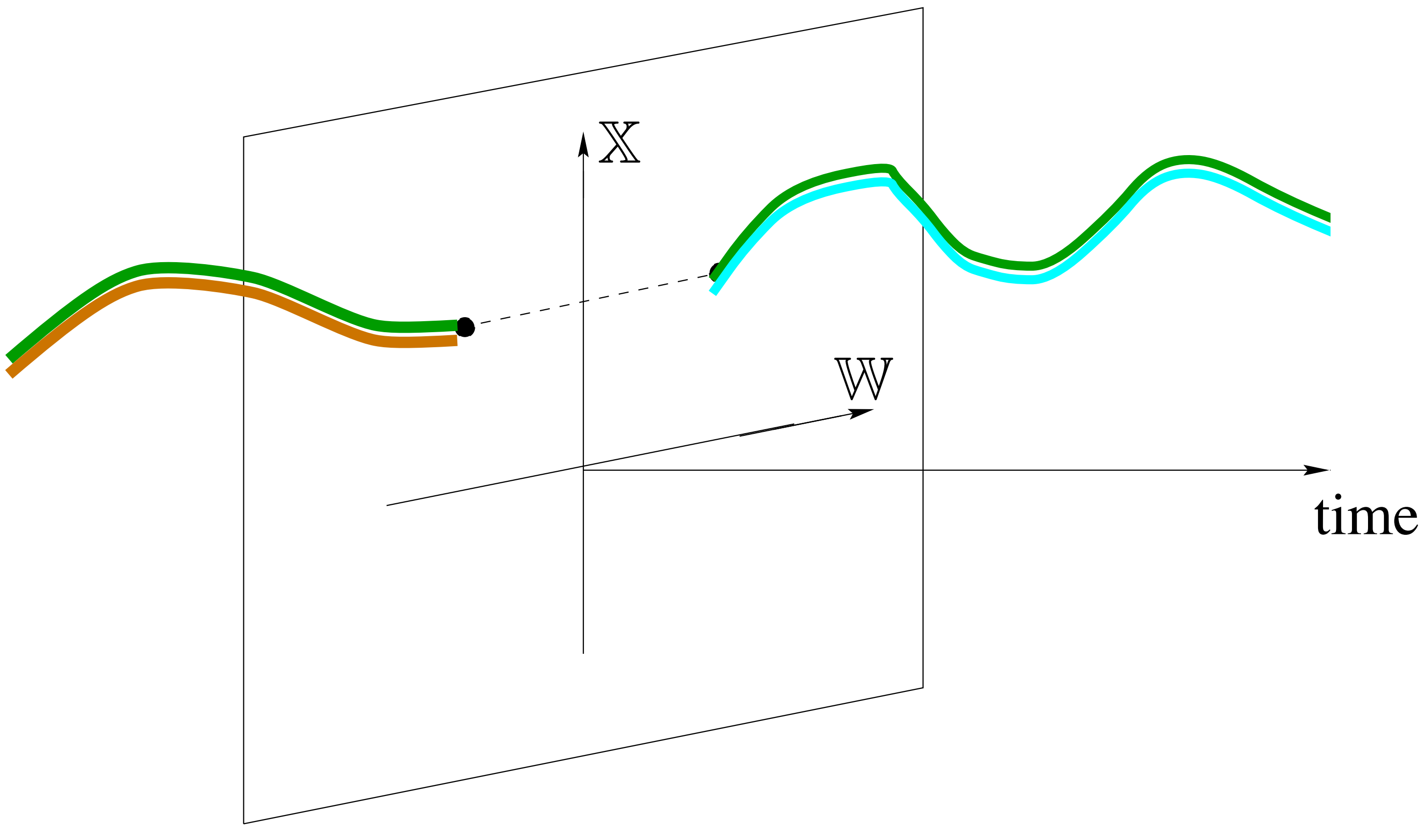
$$(w_1, x_1) \underset{t_0}{\wedge} (w_2, x_2) \in \mathfrak{B}_{\text{full}}.$$

$\underset{t_0}{\wedge}$ **denotes *concatenation* at $t_0$, defined as**

$$f_1 \underset{t_0}{\wedge} f_2(t) := \begin{cases} f_1(t) \text{ for } t < t_0 \\ f_2(t) \text{ for } t \geq t_0 \end{cases}$$

**In pictures:**






**Concatenation**

**This definition is the implementation of the idea:**

***The state at time $t$, $x(t)$, contains all the information (about $(w, x)$!) that is relevant for the future behavior.***

**The state = the memory (nothing 'minimal' implied!).**

**The past and the future are 'independent', conditioned on (given) the present state.**

## Examples of state systems:

### 1. Discrete-time systems.

**A latent variable system described by a difference equation that is *first order* in the latent variable $x$, and *zero-th order* in the manifest variable $w$:**

$$\boxed{F(x(t+1), x(t), w(t), t) = 0.}$$

## 2. Continuous-time systems.

**A latent variable system described by a differential equation that is** ***first order*** **in the latent variable** $x$**, and** ***zero-th order*** **in the manifest variable** $w$**:**

$$\boxed{F(\frac{d}{dt}x(t), x(t), w(t), t) = 0.}$$

**In particular, the ubiquitous**

$$\frac{d}{dt}x(t) = f(x(t), u(t)), y(t) = h(x(t), u(t));$$

$$w(t) = (u(t), y(t)).$$

3. Automata.

4. Trellis diagrams.

5. QM:

$$\frac{d}{dt}\psi = i\hbar H(\psi)\,, \quad p = |\psi|^2;$$

$\psi =$ the 'wave function';

$p(x,t) =$ the 'probability' density of the particle's position.

**For discrete time state systems** $\rightsquigarrow$

**Theorem: The 'complete' latent variable system**

$$\Sigma_X = (\mathbb{Z}, \mathbb{W}, \mathbb{X}, \mathfrak{B}_{\text{full}})$$

**is a state system *if and only if* $\mathfrak{B}_{\text{full}}$ admits a representation as a difference equation that is *first order* in the latent variable $x$, and *zero-th order* in the manifest variable $w$:**

$$\boxed{F(x(t+1), x(t), w(t), t) = 0.}$$

# LINEAR DIFFERENTIAL SYSTEMS

We now introduce the systems

$$\boxed{\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})}$$

that are

1. *linear* , meaning
   $((w_1, w_2 \in \mathfrak{B}) \wedge (\alpha, \beta \in \mathbb{R})) \Rightarrow (\alpha w_1 + \beta w_2 \in \mathfrak{B})$;

2. *time-invariant* , meaning
   $((w \in \mathfrak{B}) \wedge (t \in \mathbb{R})) \Rightarrow (\sigma^t w \in \mathfrak{B}))$,
   where $\sigma^t$ denotes the backwards $t-$shift;

3. *differential* , meaning
   $\mathfrak{B}$ consists of the solutions of a system of differential equations.

**In vector/matrix notation:**

$$w = \begin{bmatrix} w_1 \\ w_2, \\ \vdots \\ w_{\mathtt{w}} \end{bmatrix}, \quad R_k = \begin{bmatrix} R^k_{1,1} & R^k_{1,2} & \cdots & R^k_{1,\mathtt{w}} \\ R^k_{2,1} & R^k_{2,2} & \cdots & R^k_{2,\mathtt{w}} \\ \vdots & \vdots & \cdots & \vdots \\ R^k_{\mathtt{g},1} & R^k_{\mathtt{g},2} & \cdots & R^k_{\mathtt{g},\mathtt{w}} \end{bmatrix}.$$

**Yields**

$$\boxed{R_0 w + R_1 \frac{d}{dt} w + \cdots + R_n \frac{d^n}{dt^n} w = 0,}$$

**with** $R_0, R_1, \cdots, R_n \in \mathbb{R}^{\mathtt{g} \times \mathtt{w}}$.

**Combined with the polynomial matrix**

$$R(\xi) = R_0 + R_1\xi + \cdots + R_n\xi^n,$$

**we obtain the short notation**

$$\boxed{R(\frac{d}{dt})w = 0.}$$

**Including latent variables** $\rightsquigarrow$

$$\boxed{R(\frac{d}{dt})w = M(\frac{d}{dt})\ell}$$

**with** $R, M \in \mathbb{R}^{\bullet\times\bullet}[\xi]$**.**

## Examples:

**1. RLC-circuit:** **Case 1:** $CR_C \neq \frac{L}{R_L}$.

**Then the relation between $V$ and $I$ is**

$$(\frac{R_C}{R_L} + (1 + \frac{R_C}{R_L})CR_C\frac{d}{dt} + CR_C\frac{L}{R_L}\frac{d^2}{dt^2})V$$

$$= (1 + CR_C\frac{d}{dt})(1 + \frac{L}{R_L}\frac{d}{dt})R_C I.$$

**We have** $\mathtt{w} = 2$; $\mathtt{g} = 1$; $w = \begin{bmatrix} V \\ I \end{bmatrix}$; $R(\xi) =$

$$[\ (\tfrac{R_C}{R_L}+(1+\tfrac{R_C}{R_L})CR_C\xi+CR_C\tfrac{L}{R_L}\xi^2 \ \mid \ -1-(CR_C+\tfrac{L}{R_L})\xi-(CR_C\tfrac{L}{R_L})\xi^2\ ]$$

$$= [\ \tfrac{R_C}{R_L}\ \mid\ -1\ ] + [\ 1+\tfrac{R_C}{R_L}\ \mid\ -CR_C-\tfrac{L}{R_L}\ ]\,\xi + [\ CR_C\tfrac{L}{R_L}\ \mid -CR_C\tfrac{L}{R_L}\ ]\,\xi^2$$

## 2. Linear systems:

- The ubiquitous

$$\boxed{P(\tfrac{d}{dt})y = Q(\tfrac{d}{dt})u, \quad w = (u, y)}$$

with $P, Q \in \mathbb{R}^{\bullet\times\bullet}[\xi]$, $\det(P) \neq 0$ and, perhaps, $P^{-1}Q$ proper.

- The ubiquitous

$$\boxed{\tfrac{d}{dt}x = Ax + Bu; \quad y = Cx + Du, \quad w = (u, y).}$$

- The descriptor systems

$$\frac{d}{dt}Ex + Fx + Gw = 0.$$

**For simplicity of exposition, we assume smooth solutions.**

**Whence,** $R(\frac{d}{dt})w = 0$ **defines the system** $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ **with**

$$\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R(\tfrac{d}{dt})w = 0\}.$$

## NOTATION

$\mathfrak{L}^\bullet$ : all such systems (with any - finite - number of variables)

$\mathfrak{L}^{\mathtt{w}}$ : with $\mathtt{w}$ variables

$\mathfrak{B} = \ker(R(\frac{d}{dt}))$

$\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ (no ambiguity regarding $\mathbb{T}$, $\mathbb{W}$)

The theory (representations, elimination, controllability, observability, algorithms, control, etc.) of $\mathfrak{L}^\bullet$ is very complete.

# STATE REPRESENTATIONS

# DESCRIPTOR SYSTEMS

**Theorem: The latent variable system $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{n}, \mathfrak{B}_{\text{full}})$ with $\mathfrak{B}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{n}}$ is a state system *if and only if* $\mathfrak{B}_{\text{full}}$ admits a kernel representation that is *first order* in the latent variable $x$, and *zero-th order* in the manifest variable $w$.**

**In other words, iff there exist matrices $E, F, G \in \mathbb{R}^{\bullet \times \bullet}$ such that this kernel representation takes the form of a *descriptor system:***

$$\boxed{E\frac{d}{dt}x + Fx + Gw = 0.}$$

# MINIMALITY of STATE REPRESENTATIONS

**We can consider two types of minimality:**

1. **Minimality of the number of *equations***
2. **Minimality of the number of *state variables***

**We discuss mainly the second one.**

**Definition:** **The state system** $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{n}, \mathfrak{B}_{\mathrm{full}})$ **with** $\mathfrak{B}_{\mathrm{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{n}}$ **is said to be** ***state-minimal*** **if, whenever** $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{n}'}, \mathfrak{B}'_{\mathrm{full}})$ **with** $\mathfrak{B}'_{\mathrm{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{n}'}$ **is another state system with the same manifest behavior, there holds**

$$\mathtt{n} \leq \mathtt{n}'.$$

**One more definition...**

$\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is said to be *trim* if, for all $\mathrm{w}_0 \in \mathbb{R}^{\mathtt{w}}$, there exists $w \in \mathfrak{B}$ such that $w(0) = \mathrm{w}_0$. The state system $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{n}, \mathfrak{B}_{\text{full}})$ with $\mathfrak{B}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{n}}$ is said to be *state-trim* if, for all $\mathrm{x}_0 \in \mathbb{R}^{n}$, there exists $(w, x) \in \mathfrak{B}_{\text{full}}$ such that $x(0) = \mathrm{x}_0$.

**Theorem:**

The state system $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{n}, \mathfrak{B}_{\text{full}})$ with $\mathfrak{B}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{n}}$ is *state-minimal* iff it is *state trim* and the state $x$ is *'observable'* from $w$.

Observability :$\Leftrightarrow$ $x$ can be deduced from $w$.
I.e., $\exists\ X \in \mathbb{R}^{\mathtt{n}\times\mathtt{w}}[\xi]$ such that $(w, x) \in \mathfrak{B}_{\text{full}} \Leftrightarrow x = X(\frac{d}{dt})w$.

State-minimality $\Leftrightarrow$ the combination of trimness and observability.

**Notes:**

1. **State isomorphism theorem.** Assume $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{n}, \mathfrak{B}_{\text{full}})$ and $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{n}, \mathfrak{B}'_{\text{full}})$, $\mathfrak{B}_{\text{full}}, \mathfrak{B}'_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{n}}$ both state-minimal, same manifest behavior

   $\Rightarrow$ there exists a nonsingular $S \in \mathbb{R}^{n \times n}$ such that

   $$((w, x) \in \mathfrak{B}_{\text{full}} \text{ and } (w, x') \in \mathfrak{B}'_{\text{full}}) \Leftrightarrow (x' = Sx).$$

2. The manifest behavior is **controllable** iff a state-minimal state representation is **state-controllable** (defined in the obvious way).

3. $\exists$ algorithms on $E, F, G$ in a descriptor representation to verify its state-minimality, its equation minimality, both combined.

4. $E\frac{d}{dt}x + Fx + Gw = 0$ and $E'\frac{d}{dt}x' + F'x' + G'w = 0$ are two minimal (state- and equation-minimal) representations of the same manifest behavior iff there exist nonsingular matrices $T, S \in \mathbb{R}^{\bullet\times\bullet}$ such that

$$E' = TES, F' = TES, G' = TG.$$

All 'classical' results remain valid, except, (fortunately!) the celebrated (non-)equivalence:

state-minimality $\Leftrightarrow$ (state-observability + state-controllability).

Non-controllable systems are very 'real' and they allow state-minimal (non-controllable) state representation.

# STATE CONSTRUCTION

**Given a representation of the manifest behavior, find a (state-minimal) state representation for it.**

**Most logical : latent variable representation $\rightsquigarrow$ state representation. However, it is most convenient to discuss kernel representations first.**

**Let $X(\xi) \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$. The map $X(\frac{d}{dt})$ is called a *state map* for $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ if the full behavior**

$$\mathfrak{B}_{\mathrm{full}} = \{(w, x) \mid w \in \mathfrak{B} \text{ and } x = X(\frac{d}{dt})w\}$$

**satisfies the axiom of state. *Minimal* state map: obvious.**

**In a state-minimal representation, $x$ is always determined by a state map (because of observability), whence state maps exist.**

## CONSTRUCTION of STATE MAPS

**Define the *'shift-and-cut'* operator $\sigma$ on $\mathbb{R}[\xi]$ as follows:**

$$\sigma : p_0 + p_1\xi + \cdots + p_{n-1}\xi^{n-1} + p_n\xi^n \mapsto p_1 + p_2\xi + \cdots + p_{n-1}\xi^{n-2} + p_n\xi^{n-1}$$

**Extend-able in the obvious term-by-term way to $\mathbb{R}^{\bullet\times\bullet}[\xi]$.**

**Repeated use of the cut-and-shift on $P \in \mathbb{R}^{\bullet\times\bullet}$ yields the *'stack' operator* $\Sigma_P$, defined by**

$$\Sigma_P := \begin{bmatrix} \sigma(P) \\ \sigma^2(P) \\ \vdots \\ \sigma^{\text{degree}(P)}(P) \end{bmatrix}$$

## FROM KERNEL REPRESENTATION to STATE MAP

**There is a construction (elegant in its simplicity) of a state map in terms of the cut-and-shift and stack operators!**

**Theorem: Let $R(\frac{d}{dt})w = 0$ be a kernel representation of $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$. Then $\Sigma_R(\frac{d}{dt})$ is a state map for $\mathfrak{B}$.**

**The resulting state representation**

$$R(\frac{d}{dt})w = 0\,; \quad x = \Sigma_R(\frac{d}{dt})w$$

**need not be minimal. It is trivially state-observable, but it may not be state-trim. Using Gröbner basis techniques it can be trimmed, leading to a minimal state representation.**

## SINGLE INPUT - SINGLE OUTPUT SYSTEMS

**Apply this to**

$$p(\frac{d}{dt})y = q(\frac{d}{dt})u$$

**with**

$$\begin{aligned} p(\xi) &= p_0 + p_1\xi + \cdots + p_{n-1}\xi^{n-1} + p_n\xi^n, \quad p_n \neq 0 \\ q(\xi) &= q_0 + q_1\xi + \cdots + q_{n-1}\xi^{n-1} + q_n\xi^n \end{aligned}$$

**The cut-and-shift and stack operators yield the polynomial matrix**

$$X(\xi) = \begin{bmatrix} p_1+\cdots+p_{n-1}\xi^{n-2}+p_n\xi^{n-1} & -q_1-\cdots-q_{n-1}\xi^{n-2}-q_n\xi^{n-1} \\ p_2+\cdots+p_{n-1}\xi^{n-3}+p_n\xi^{n-2} & -q_2-\cdots-q_{n-1}\xi^{n-3}-q_n\xi^{n-2} \\ \vdots & \vdots \\ p_{n-1}+p_n\xi & -q_{n-1}-q_n\xi \\ p_n & -q_n \end{bmatrix}$$

It follows that $x = X(\frac{d}{dt})$ is a state map, in fact, a state minimal one, even if the system is not controllable, i.e., if $p$ and $q$ have a common factor.

Minimal state map $\cong$ basis for span of the rows of $X$ modulo $R$.

Number the rows of $X$ in reverse order. A small calculation shows that this choice of the state variables leads to the so-called *observer canonical form*, the i/s/o representation

$$A = \begin{bmatrix} -p_0/p_n & 1 & 0 & \cdots & 0 & 0 \\ -p_1/p_n & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -p_{n-1}/p_n & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} q_0 - p_0 q_n/p_n \\ q_1 - p_1 q_n/p_n \\ \vdots \\ q_{n-1} - p_{n-1} q_n/p_n \end{bmatrix},$$

$$C = \begin{bmatrix} 1/p_n & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} q_n/p_n \end{bmatrix}.$$

**Another immediate choice is to pick linear combinations of the rows of $X$ so that the resulting state map matrix takes the form**

$$X'(\xi) = \begin{bmatrix} 1 & \star \\ \xi & \star \\ \vdots & \vdots \\ \xi^{n-2} & \star \\ \xi^{n-1} & \star \end{bmatrix}$$

**The $\star$'s are obtained from the polynomial $b(\xi) \in \mathbb{R}[\xi]$ defined by the equation**

$$p(\xi)b(\xi^{-1}) = q(\xi) \quad (\text{ modulo } \xi^{-1}\mathbb{R}[\xi^{-1}]).$$

**Then**

$$X'(\xi) = \begin{bmatrix} 1 & b_0 \\ \xi & b_1 + b_0\xi \\ \vdots & \vdots \\ \xi^{n-2} & b_{n-2} + b_{n-3}\xi + \cdots + b_0\xi^{n-2} \\ \xi^{n-1} & b_{n-1} + b_{n-2}\xi + \cdots + b_0\xi^{n-1} \end{bmatrix}.$$

This leads to the *observable canonical form* , the i/s/o representation

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\frac{p_0}{p_n} & -\frac{p_1}{p_n} & -\frac{p_2}{p_n} & \cdots & -\frac{p_{n-1}}{p_n} \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} b_0 \end{bmatrix}.$$

Image representations + state construction $\rightsquigarrow$

*controller canonical form* and *controllable canonical form* .

**Notes:**

- **Basic idea of algorithms:**
  **from latent variable representation directly to state model.**
  **This complements the existent algorithms**
  **transfer function → i / s / o representation;**
  **impulse response → i / s / o representation.**
- **Our state construction is easily extended to state / input construction.**

- **Examples of useful special (minimal) state representations:**

  **i/s/o representation:**

$$\frac{d}{dt}x = Ax + Bu, \;\; y = Cx + Du, w = (u, y),$$

  **output nulling representation:**

$$\frac{d}{dt}x = Ax + Bw, \;\; 0 = Cx + Dw,$$

  **driving variable representation:**

$$\frac{d}{dt}x = Ax + Bv, \;\; w = Cx + Dv.$$

- **Immediately deduced from descriptor representation:**

$$E\frac{d}{dt}x + Fx + Gw = 0.$$

# BALANCED REALIZATIONS

## FROM KERNEL REPR. TO BALANCED REAL.

**For simplicity, we discuss only single input - single output systems**

$$p(\frac{d}{dt})y = q(\frac{d}{dt})u$$

**assumed controllable, i.e., with $p$, $q$ co-prime,**

$\mathtt{n} = \text{degree(p)} \geq \text{degree(q)}$,

**and stable, i.e., $p$ Hurwitz.**

**Problem: Pass from $p$, $q$ to a balanced state representation using polynomial algebra.**

**Computation of the controllability and observability Gramians (two-variable polynomials):**

$$C(\zeta, \eta) = \frac{p(\zeta)p(\eta) - p(-\zeta)p(-\eta)}{\zeta + \eta},$$

$$O(\zeta, \eta) = \frac{q(\zeta)q(\eta) - x(\zeta)p(\eta) - p(\zeta)x(\eta)}{\zeta + \eta},$$

**where $x \in \mathbb{R}[\xi]$ is given by the Bezout equation**

$$x(-\xi)p(\xi) + p(-\xi)x(\xi) = q(-\xi)q(\xi).$$

Factor $C, O$ as

$$C(\zeta,\eta) = \sum_{k=1}^{n} \sigma_k^{-1} x_k(\zeta) x_k(\eta),$$

$$O(\zeta,\eta) = \sum_{k=1}^{n} \sigma_k \, x_k(\zeta) x_k(\eta),$$

with $\sigma_1 \geq \sigma_2 \geq \cdots \sigma_n > 0$. Solve for $k = 1, \ldots, n$, the equations

$$\xi x_k(\xi) = \sum_{k'=1}^{n} a_{k,k'} x_{k'}(\xi) + b_k p(\xi),$$

$$q(\xi) = \sum_{k'=1}^{n} c_{k'} x_{k'}(\xi) + d p(\xi).$$

Then the matrices with entries $a_{k,k'}$, $b_k$, $c_k$, $d$ define a balanced realization. The balanced reduction is obvious from here.

**Open Problem:**

**Algorithm**

$$(p, q) \quad \rightsquigarrow \quad (p_{\text{balred}}, q_{\text{balred}}).$$

**AAK version of this: cfr. Fuhrmann's book.**

**Goal: algorithms passing from image, latent variable representation to balanced realization.**

# STATE for PDE's

## BEHAVIORS of n-D SYSTEMS

*A system* **:=** $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$

$\mathbb{T}$ = **the set of *independent* variables**
**time, space, time and space**

$\mathbb{W}$ = **the set of *dependent* variables (= where the variables take on their values), signal space, space of field variables, . . .**

$\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$ : ***the behavior*** **= the admissible trajectories**

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$$

**for a trajectory** $w : \mathbb{T} \to \mathbb{W}$, **we thus have:**

$w \in \mathfrak{B}$ **: the model allows the trajectory** $w$,
$w \notin \mathfrak{B}$ **: the model forbids the trajectory** $w$.

**In this section,** $\mathbb{T} = \mathbb{R}^n$, **('n-D systems')**
$\mathbb{W} = \mathbb{R}^{\mathtt{w}}$,
$w : \mathbb{R}^n \to \mathbb{R}^{\mathtt{w}}, (w_1(x_1, \cdots, x_{\mathtt{n}}), \cdots, w_{\mathtt{w}}(x_1, \cdots, x_{\mathtt{n}})),$
**often,** $\mathtt{n} = 4$, **independent variables** $(t, x, y, z)$,
$\mathfrak{B}$ = **solutions of a system of constant coefficient linear PDE's.**

**'Linear distributed differential systems'.**

# n-D LINEAR DIFFERENTIAL SYSTEMS

$\mathbb{T} = \mathbb{R}^n$,
$\mathbb{W} = \mathbb{R}^{\mathtt{w}}$,
$\mathfrak{B} =$ the solutions of a linear constant coefficient system of PDE's.

**Let** $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$, **and consider**

$$R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = 0 \quad (*)$$

**Define**

$$\mathfrak{B} := \{w \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}}) \mid (*) \text{ holds } \}$$

$\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$ **mainly for convenience.**

**Example:** ***Maxwell's equations***

$$\begin{aligned}
\nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho\,, \\
\nabla \times \vec{E} &= -\frac{\partial}{\partial t}\vec{B}, \\
\nabla \cdot \vec{B} &= 0\,, \\
c^2 \nabla \times \vec{B} &= \frac{1}{\varepsilon_0}\vec{j} + \frac{\partial}{\partial t}\vec{E}.
\end{aligned}$$

$\mathbb{T} = \mathbb{R} \times \mathbb{R}^3$ **(time and space),**

$w = (\vec{E}, \vec{B}, \vec{j}, \rho)$

**(electric field, magnetic field, current density, charge density),**

$\mathbb{W} = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$,

$\mathfrak{B}$ **= set of solutions to these PDE's.**

**Note: 10 variables, 8 equations!** $\Rightarrow \exists$ **free variables.**

**Maxwell's equation for the electrical variables (with $\vec{B}$ 'eliminated'):**

$$\begin{aligned} \nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho\,, \\ \varepsilon_0 \frac{\partial}{\partial t}\nabla \cdot \vec{E} + \nabla \cdot \vec{j} &= 0, \\ \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{E} + \varepsilon_0 c^2 \nabla \times \nabla \times \vec{E} + \frac{\partial}{\partial t}\vec{j} &= 0. \end{aligned}$$

## MARKOVIAN n-D SYSTEMS

**What is the notion of state for such systems?**

**What does 'Markov' mean?**

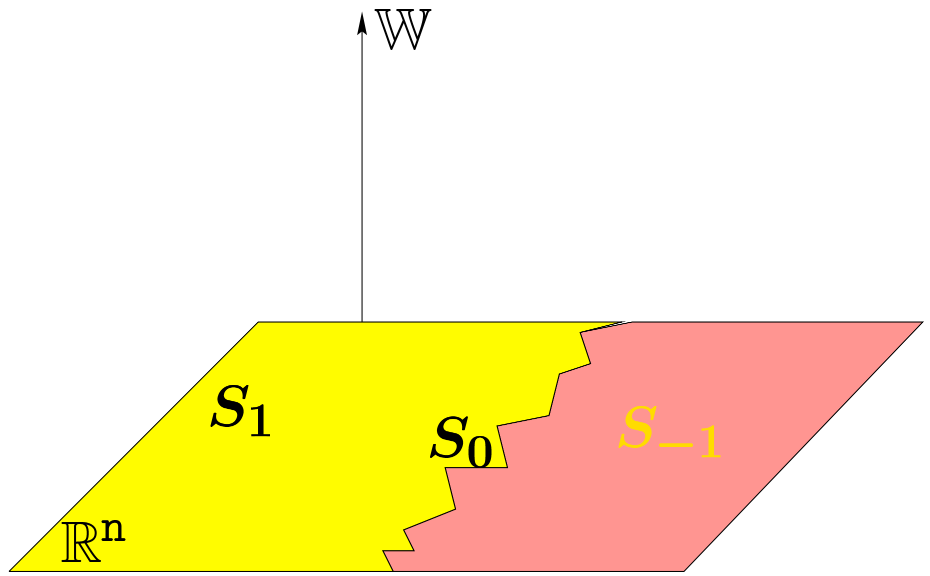
$\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$ **is said to be Markovian if for all nice partitions of** $\mathbb{R}^n = S_{-1} \cup S_0 \cup S_1$ **there holds:**

$$w_1, w_2 \in \mathfrak{B} \quad \text{and} \quad {w_1}_{|S_0} = {w_2}_{|S_0}$$

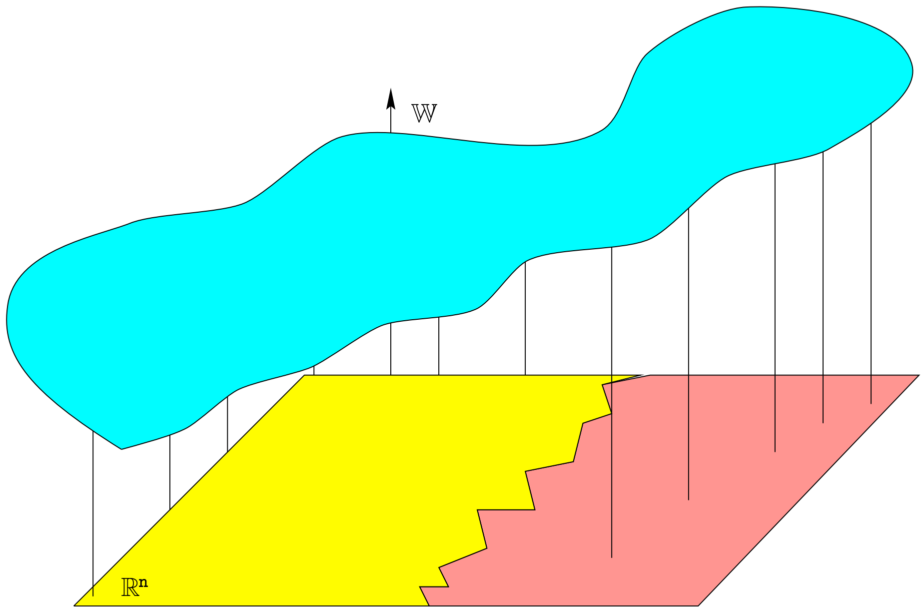
$$\Longrightarrow$$

$$w_1 \underset{S_0}{\wedge} w_2 \in \mathfrak{B}.$$

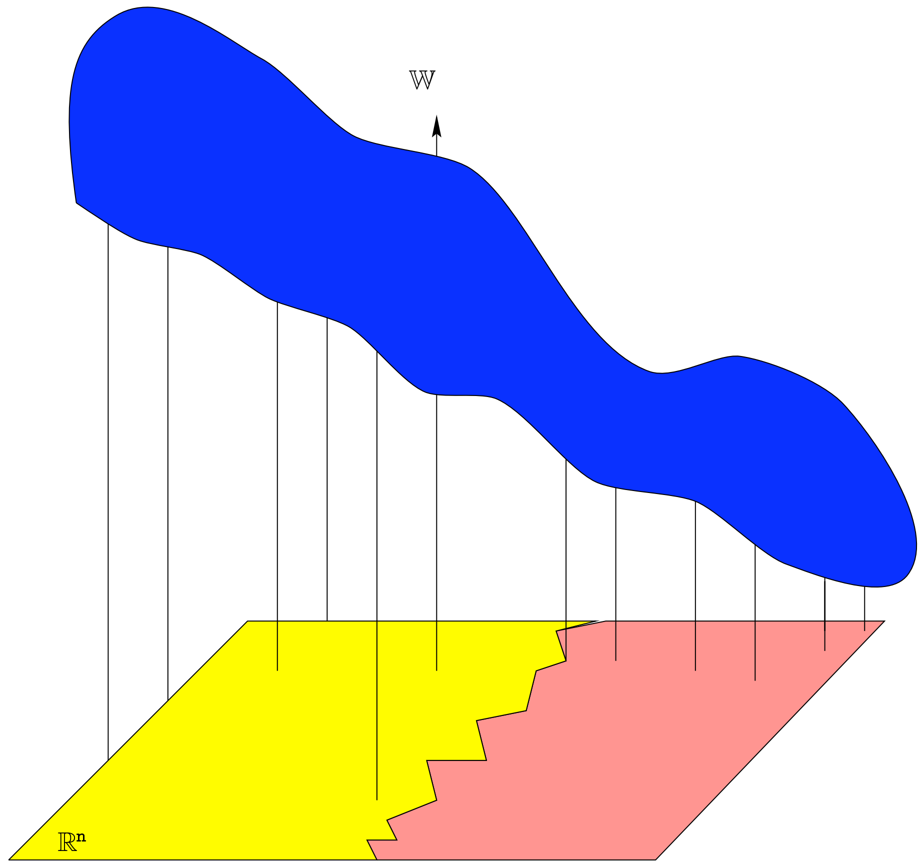
**($\underset{S_0}{\wedge}$ := 'concatenation' at $S_0$).**



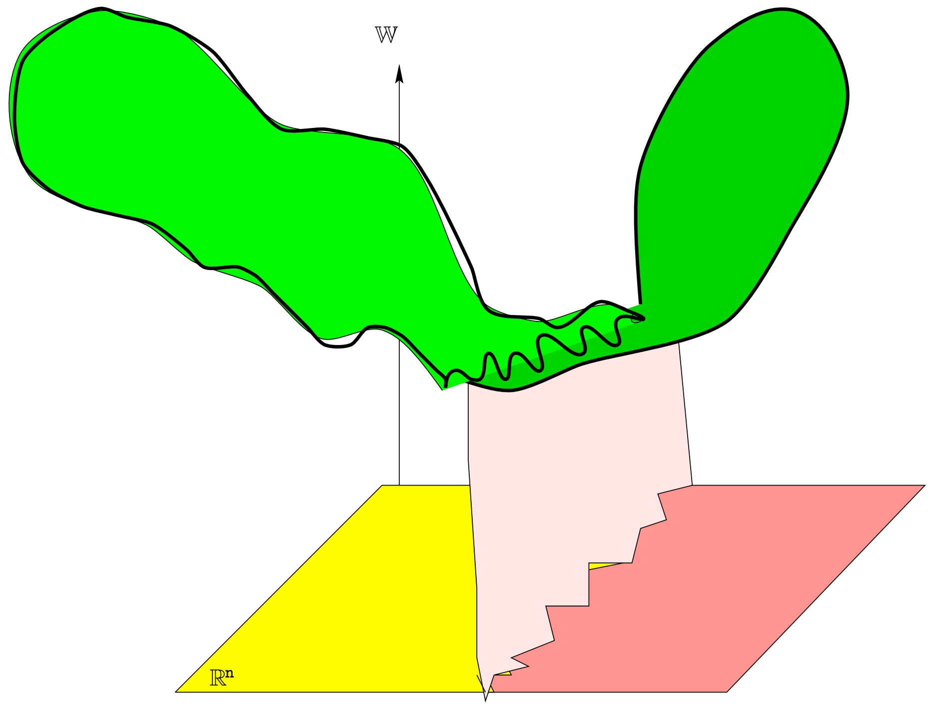

**A nice partition**




**The first solution**




**The second solution**




**The concatenated solution**

## CONJECTURE

$\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$ is Markovian if and only if

$$\mathfrak{B} = \ker(R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})),$$

with $R$ first order, i.e.,

$$R(\xi_1, \cdots, \xi_{\mathtt{n}}) = R_0 + R_{1,1}\xi_1 + R_{1,2}\xi_2 + \cdots + R_{1,\mathtt{n}}\xi_{\mathtt{n}}.$$

"If"-part is clear; "only if"-part is the problem.

Example: Maxwell's equations, $\vec{B}$ induces state representation of electrical behavior. Not observable, thou.

## CONCLUDING REMARKS

- **A system $\cong$ a manifest behavior**
- **First principles models $\rightsquigarrow$ systems with latent variables**
- **State systems: latent variable systems in which the state 'splits' the past and the future**
- **State construction: for linear differential systems via cut-and-shift map and Gröbner basis algorithms**
- **Balanced reduction via polynomial algebra**
- **Conjecture for PDE's: Markovian $\Leftrightarrow$ first order**

**Thank you!**